UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél.:(1)39 63 55 11

Rapports de Recherche

N°1743

Programme 2

*Calcul symbolique, Programmation
et Génie logiciel*

FAST COMPUTATION OF SOME ASYMPTOTIC FUNCTIONAL INVERSES

Bruno SALVY

Septembre 1992

Fast Computation of Some Asymptotic Functional Inverses

Bruno Salvy

Abstract

G. Robin showed that in several naturally occurring asymptotic expansions of the form

$$y(x) \approx \sum_{n\geq 0} \frac{P_n(\log\log x)}{\log^n x},$$

the polynomials P_n satisfy a simple relation

$$P'_{n+1} = aP'_n + (bn+c)P_n.$$

These results do not give a way to compute these polynomials, since the constant term remains undetermined by this equation. In this note, we give a new derivation of some of Robin's results, and show how the constant terms can be computed with only manipulations of one-variable formal power series. From there, all the P_n can be computed efficiently.

Calcul rapide de certains inverses fonctionnels asymptotiques

Résumé

G. Robin a montré que dans plusieurs développements asymptotiques courants de la forme

$$y(x) \approx \sum_{n\geq 0} \frac{P_n(\log\log x)}{\log^n x},$$

les polynômes P_n vérifient une relation simple

$$P'_{n+1} = aP'_n + (bn+c)P_n.$$

Ces résultats ne permettent pas de calculer ces polynômes, puisque cette équation ne détermine pas le terme constant. Dans cette note, nous donnons une nouvelle preuve de certains des résultats de Robin, et nous montrons comment on peut réduire le calcul des termes constants à des manipulations de séries formelles en une variable. Partant de là, tous les P_n peuvent être calculés efficacement.

Fast computation of some asymptotic functional inverses

Bruno Salvy

Abstract

G. Robin showed that in several naturally occurring asymptotic expansions of the form

$$y(x) \approx \sum_{n\geq 0} \frac{P_n(\log\log x)}{\log^n x},$$

the polynomials P_n satisfy a simple relation

$$P'_{n+1} = aP'_n + (bn+c)P_n.$$

These results do not give a way to compute these polynomials, since the constant term remains undetermined by this equation. In this note, we give a new derivation of some of Robin's results, and show how the constant terms can be computed with only manipulations of one-variable formal power series. From there, all the P_n can be computed efficiently.

Introduction

Consider the formal equation

$$e^y y^{-\alpha} \sum_{n\geq 0} \frac{d_n}{y^n} = x, \tag{1}$$

where $\alpha \neq 0$ and $d_0 \neq 0$. We are interested in finding a formal asymptotic expansion of $y(x)$ as x tends to infinity. Set $D(u) = \sum d_n u^n$, then rewriting this equation as

$$y = \log x + \alpha \log y - \log D(1/y), \tag{2}$$

and considering the *iteration process* $y_{n+1} = H(y_n)$ where $H(y)$ is the right hand side of Equation (2) and $y_0 = \log x$, we get

$$y_0 = \log x, \quad y_1 = \log x + \alpha\log\log x - \log d_0 + \cdots,$$

$$y_2 = \log x + \alpha\log\log x - \log d_0 + \frac{\alpha^2\log\log x - \alpha\log d_0 - d_1/d_0}{\log x} + \cdots, \ldots$$

It can be shown (see [6]) that this process converges to a formal expansion

$$y(x) \approx \log(x) + \sum_{n\geq 0} \frac{P_n(\log\log x)}{\log^n x}, \tag{3}$$

where the P_n are polynomials of degree n for $n > 0$, and P_0 is of degree 1.

From the computational point of view, it is interesting to note that none of the current computer algebra systems can iterate this process far without help, some of them by lack of any built-in feature to handle asymptotic expansions involving simultaneously $\log x$ and $\log\log x$, others because they do not simplify intermediate results and therefore manipulate huge expressions, eating up memory and computation time.

A first step toward a more efficient process to compute the polynomials P_n was achieved by G. Robin in [9]. He showed that these polynomials satisfy a simple relation that does not depend on $\{d_n\}$. More precisely, one has

$$P'_{n+1} = \alpha(P'_n - nP_n), \quad n \geq 0. \tag{4}$$

Besides [9], this relation is more or less preserved by simple operations on $y(x)$: functions like $\log y$, y^β or $\exp y$ all have an asymptotic expansion involving polynomials that satisfy a similar recurrence equation.

However, these relations do not give a way to compute the P_n since they do not determine their constant terms. In this note, we give a new proof of some of the results of [9], and describe a simple way to compute the constant terms. Then we show how the expansions of y, $\log y$, y^β and $\exp y$ can be computed by elementary manipulations of *one-variable* power series.

The approach we take is to view the asymptotic expansion as the *generating function* of the P_n: setting $\zeta = \log\log x$ and $t = 1/\log x$, we consider

$$P(\zeta, t) = \sum_{n\geq 0} P_n(\zeta)t^n.$$

Then we translate the functional equation of y into a functional equation for this generating function. From this we can compute both the recurrence of the P_n and the generating function for their constant terms. The recurrence is obtained by forming a *partial differential equation* for $P(\zeta, t)$:

$$\left(\frac{1}{\alpha} - t\right)\frac{\partial P}{\partial \zeta} + t^2\frac{\partial P}{\partial t} = 1, \tag{5}$$

which is then translated into (4) at the level of the coefficients. The generating function for the constant terms is seen to satisfy a simple functional equation, derived by setting $\zeta = 0$ in the functional equation for $P(\zeta, t)$. From this equation, the constant terms can be computed independently of the polynomials. This is summarized in Theorem 1 of Section 1. While this yields an analogue to Lagrange's inversion theorem for power series, Section 2 deals with an analogue to the Lagrange-Bürmann theorem. Although not all functions $F(y)$ have such an asymptotic expansion, it is shown there that when $F(y)$ is $\log y$ or of the form $\exp(\beta y)y^\gamma G(1/y)$, with G a formal power series, then the generating function for $F(y)$ satisfy a partial differential equation induced by (5), while the generating function of the constant terms is easily computed. In Section 3 we give a few examples taken from [9] and from the references therein. Section 4 is devoted to complexity concerns.

1 Inversion

In order to get a functional equation for $P(\zeta, t)$, we first transform Equation (2) by setting $y - \log x = P(\zeta, t)$:

$$P = \alpha\zeta + \alpha\log(1 + tP) - \log D\left(\frac{t}{1 + tP}\right). \tag{6}$$

Since formally tP is of order $\log\log x/\log x$ and thus tends to 0, when we iterate this equation and arrange the terms in increasing powers of t, only manipulations of formal power series in t are involved and ζ appears as a parameter. In other words, we have decoupled $\log x\,(= 1/t)$ and $\log\log x\,(= \zeta)$.

Now, differentiating Equation (6) with respect to ζ and t, we get two equations involving $(D'/D)[t/(1 + tP)]$. Eliminating this term and rearranging the equation we get the partial differential equation (5). Taking the coefficient of t^n in (5) we get:

$$\frac{P'_{n+1}}{\alpha} = P'_n - nP_n, \quad n \geq 0, \quad \text{and} \quad P'_0 = \alpha. \tag{7}$$

On the other hand, setting $\zeta = 0$ in (6), we also get the generating function for the constant terms:

$$\mathcal{P}_0(t) := P(0,t) = \sum_{n\geq 0} P_n(0)t^n,$$

which satisfies

$$\mathcal{P}_0 = \alpha \log(1 + t\mathcal{P}_0) - \log D\left(\frac{t}{1+t\mathcal{P}_0}\right). \tag{8}$$

Thus we have proved the following theorem, where the notation $[t^k]f(t)$ denotes the kth coefficient in the Taylor expansion of f at the origin.

Theorem 1 *Let y satisfy $e^y y^{-\alpha} D(1/y) = x$, with D a formal power series, $\alpha \neq 0$ and $D(0) \neq 0$. Then the formal asymptotic expansion of y as x tends to infinity is*

$$y(x) \approx \log x + \sum_{n\geq 0} \frac{P_n(\log\log x)}{\log^n x},$$

where the P_n are polynomials that can be computed as follows:

1. *Iterate n times (8):*
$$u_{k+1}(t) = \alpha \log[1 + tu_k(t)] - \log D\left(\frac{t}{1+tu_k(t)}\right)$$
from $u_0 = -\log D(0)$, which gives $P_k(0) = [t^k]u_n(t)$ for $k \in \{0, \ldots, n\}$.
2. *Iterate $P'_{n+1} = \alpha(P'_n - nP_n)$ from $P_0 = \alpha\zeta - \log D(0)$ to get the other coefficients.*

While computer algebra systems could not iterate Equation (2), Equation (8) is much easier, since it is reduced to one-variable formal power series manipulations. For instance, we get with Maple

$$P_0(0) = -\ln(d_0), \quad P_1(0) = -\alpha\ln(d_0) - d_1/d_0,$$

$$P_2(0) = \frac{d_1^2}{2d_0^2} - \frac{d_2 + (\alpha + \ln(d_0))d_1}{d_0} - \alpha/2\ln^2(d_0) - \alpha^2\ln(d_0), \ldots$$

We note the following useful corollary from [4], which Comtet proved by Lagrange inversion.

Corollary 1 (Comtet) *The asymptotic expansion of the solution of $e^y y^{-\alpha} = x$ is of the form (3), the polynomials P_n being given by*

$$P_n(\zeta) = \alpha^{n+1} \sum_{k=1}^{n} s_{n,n-k+1} \frac{\zeta^k}{k!},$$

where $s_{n,k}$ are signed Stirling numbers of the first kind.

In particular, the fact that $\mathcal{P}_0$ is 0, together with Equation (7) yields an efficient program to compute the asymptotic expansion to any order of Maple's W function, which corresponds to the case $\alpha = -1$, and is also related to the generating function of Cayley trees. Currently Maple can only expand W to the order 10.
Proof. In the case when $D = 1$, considering the valuation of $\mathcal{P}_0$ in Equation (8) shows that $\mathcal{P}_0 = 0$. At the level of the coefficients of P_n:

$$P_n(\zeta) = \sum_{k=0}^{n} u_{n,k},$$

this means that $u_{n,0} = 0$ for all n. We also have from Theorem 1 that $u_{n,n+1} = 0$ for all n, except for $u_{0,1} = \alpha$. This gives the boundary conditions for the recurrence we deduce from (7):

$$\frac{(k+1)}{\alpha} u_{n+1,k+1} = (k+1)u_{n,k+1} - nu_{n,k},$$

valid for all $k \geq 0$ and $n \geq 0$. It is natural to simplify this recurrence by setting $v_{n,k} = \alpha^{-n-1} k! u_{n,k}$:

$$v_{n+1,k+1} = v_{n,k+1} - n v_{n,k}.$$

Now, this is almost the classical recurrence for Stirling numbers of the first kind (see [5, ch. V]), which is reached by setting $v_{n,k} = s_{n,n-k+1}$. The initial conditions deduced from those of u prove that these are really the Stirling numbers of the first kind. □

From this proof, we also deduce that in all cases (even if the formal power series D is not 1), if we set

$$P_n(\zeta) = \alpha^{n+1} \sum_{k=0}^{n} w_{n,n-k+1} \frac{\zeta^k}{k!},$$

then $w_{n,k}$ satisfies the recurrence for Stirling numbers of the first kind

$$w_{n+1,k+1} = w_{n,k} - n w_{n,k+1},$$

with initial conditions $w_{0,0} = 1$ and $w_{n,0} = 0$ for $n > 0$. The other initial conditions $w_{0,n}$, and only them, are determined by D.

2 Extensions

We consider the effect of the change of variable

$$f(P_0, P, t) = F(\zeta, t),$$

with $P(\zeta, t)$ and $P_0(\zeta) = \alpha\zeta - \ln(d_0)$ defined as before. Differentiating with respect to ζ and t we obtain two partial differential equations. We then eliminate the partial derivatives of P using Equation (5) and get

$$\left(\frac{1}{\alpha} - t\right) \frac{\partial F}{\partial \zeta} + t^2 \frac{\partial F}{\partial t} = \frac{\partial f}{\partial P} + (1 - \alpha t) \frac{\partial f}{\partial P_0} + t^2 \frac{\partial f}{\partial t}. \tag{9}$$

From this we deduce the results summarized in the following.

Theorem 2 *Let y satisfy $e^y y^{-\alpha} D(1/y) = x$, then the following asymptotic expansions hold with $\mathcal{P}_0(t)$ given by Theorem 1:*

$$\log y \approx \log\log x + \sum_{n\geq 0} \frac{Q_n(\log\log x)}{\log x^n},$$

with $Q_0 = 0$, $Q'_{n+1}/\alpha = Q'_n - nQ_n$ and the generating function of $\{Q_n(0)\}$ is $\log(1 + t\mathcal{P}_0(t))$.

$$e^{\beta y} y^\gamma G(1/y) \approx \left(\frac{x}{d_0}\right)^\beta (\log x)^{\alpha\beta+\gamma} \sum_{n\geq 0} \frac{Q_n(\log\log x)}{\log x^n},$$

for any formal power series G with nonzero constant term. Here, $Q_0 = G(0)$, $Q'_{n+1}/\alpha = Q'_n + (\alpha\beta+\gamma-n)Q_n$ and the generating function of $\{Q_n(0)\}$ is

$$e^{\beta(\mathcal{P}_0(t)+\log d_0)}(1 + t\mathcal{P}_0(t))^\gamma G\left(\frac{t}{1 + t\mathcal{P}_0(t)}\right).$$

Proof. The first expansion follows from rewriting y as $t^{-1}(1+tP)$ and then applying (9) with $f = \log(1+tP)$. For the second one, we write y as $\log x + P_0 + (P - P_0)$, and then consider $f = \exp(\beta(P - P_0))(1+tP)^\gamma G[t/(1+tP)]$. □

Note that this theorem cannot be much generalized: simple functions like $y \log y$ do not admit a nice asymptotic expansion of this form.

3 Applications

This section reviews some results from [9] and references therein. In all cases, we give the generating function for the constant coefficients. As a consequence, we get a fast algorithm to compute all these asymptotic expansions.

The following is a special case of Theorem 2, when $D = 1$ (see also [4]).

Corollary 2 (Comtet) *The asymptotic expansion of y subject to $y/\log^\alpha y = x$ is:*

$$y \approx x \log^\alpha x \sum_{n \geq 0} \frac{Q_n(\log\log x)}{\log^n x}, \quad x \to \infty,$$

with $Q'_{n+1}/\alpha = Q'_n + (\alpha - n)Q_n$, $Q_n(0) = 0$ for $n > 0$ and $Q_0(0) = 1$.

Proof. We first set $y = e^Y$, and then apply the last formula of Theorem 2, using the fact already noted in Corollary 1, that in this case $P_n(0) = 0$. □

Our next corollary is related to a result of Cipolla ([3]).

Corollary 3 *The asymptotic expansion of the kth prime number is*

$$p_k \approx k \log k \sum_{n \geq 0} \frac{Q_n(\log\log k)}{\log^n k},$$

with $Q_0 = 1$, $Q'_{n+1} = Q'_n + (1-n)Q_n$ and the generating function of $\{Q_n(0)\}$ is $\exp(U(t))$, U being defined by

$$U = \log(1 + tU) - \log \mathcal{E}\left(\frac{t}{1+tU}\right),$$

$\mathcal{E}(x) = \sum n!x^n$ being the ordinary generating function of $n!$.

Proof. This results from the fact (see, e.g. [1]) that the number $\pi(x)$ of primes less than x is $\mathrm{Li}(x) + O(R(x))$, where Li is the logarithmic integral, whose asymptotic expansion is

$$\mathrm{Li}(x) \approx \frac{x}{\log x}\left[1 + \frac{1!}{\log x} + \frac{2!}{\log^2 x} + \cdots\right], \tag{10}$$

and $R(x)$ depends on Riemann hypothesis. What happens is that (even without the hypothesis), the expansion of p_k depends only on the expansion of Li. Thus we have to invert (10). As previously, this is achieved by setting $y = e^Y$, and then appealing to Theorem 2. □

Our next corollary gives the constant terms in two asymptotic expansions discovered in [8]. The function $g(n)$ is defined as the maximal order of the permutations in S_n.

Corollary 4 *The following asymptotic expansion holds:*

$$\log g(n) \approx \sqrt{n \log n} \sum_{k \geq 0} \frac{Q_k(\log\log n)}{\log^k n},$$

with $Q_0 = 1$, $Q'_{k+1} = Q'_k + (1-k)Q_k$ and the generating function of $\{Q_k(0)\}$ is $\exp(U(t)/2)$, with U defined in Corollary 3. The number of prime factors of $g(n)$ satisfies asymptotically

$$\omega(g(n)) \approx 2\sqrt{\frac{n}{\log n}} \sum_{k \geq 0} \frac{Q_k(\log\log n)}{\log^k n},$$

where $Q_0 = 1$, $Q'_{n+1} = Q'_n - (n + 1/2)Q_n$ and the generating function of $\{Q_n(0)\}$ is

$$\frac{\exp(U(t)/2)}{1 - tU(t)} \mathcal{E}\left(\frac{2t}{1 + tU(t)}\right),$$

with U and $\mathcal{E}$ as in Corollary 3.

Proof. This results from a theorem of [8] stating that for $a > 0$,

$$\begin{aligned} \log g(n) &= \sqrt{\mathrm{Li}^{-1}(n)} + O(\sqrt{n} e^{-a\sqrt{\log n}}), \\ \omega(g(n)) &= \mathrm{Li}(\sqrt{\mathrm{Li}^{-1}(n)}) + O(\sqrt{n} e^{-a\sqrt{\log n}}). \end{aligned}$$

In the first case, we set $x = e^Y$ in (10), and then apply the last case of Theorem 2 with $\beta = 1/2$, $\gamma = 0$ and $G = 1$. The second case is obtained by setting $\beta = 1/2$, $\gamma = -1$, and $G = \mathcal{E}(2y)$. $\square$

4 Complexity estimates

Once the generating function for the constant terms has been computed, the polynomials are obtained by iterating

$$P_{n+1} = P_n + (an + b) \int P_n,$$

from P_0 a polynomial of degree less than 1. Thus, computation of the $(n+1)$st polynomial from the nth one requires $2n$ multiplications and n additions in the coefficient field. As a consequence, the polynomials up to the nth are computed with n^2 multiplications and $n(n-1)/2$ additions in the coefficient field. The order n^2 seems unavoidable since there are $n^2/2$ coefficients to compute.

Computation of the constant terms turns out to be the most expensive part of the process. We first have to compute the iteration of Theorem 1. Each of the n iteration steps requires substitution of a formal power series of length n into a formal power series of length n. Without any supplementary information on D, this can be achieved in $O(n^{5/2}\sqrt{\log n})$ coefficient operations, or $O(n^{3/2} \log^{3/2} n)$ if FFT is used (see [2, 7]). Since this is done n times, we get the generating function $\mathcal{P}_0$ of the constant terms in $O(n^{7/2}\sqrt{\log n})$ coefficient operations. This series has to be substituted in Theorem 2, into a series which can be computed in $O(n^{5/2}\sqrt{\log n})$, the substitution requiring $O(n^{5/2}\sqrt{\log n})$ coefficients operations.

As a conclusion, in all our theorems and corollaries, *all the polynomials up to the nth one can be computed in $O(n^{7/2}\sqrt{\log n})$ coefficient operations, the cost being dominated by the computation of the constant terms.* If the formal power series D and G satisfy "suitable" differential equations, then the complexity can be lowered to $O(n^3)$ coefficient operations (see [2]).

In practice, we want to compare this algorithm with the simpler one outlined in our introduction that consists in iterating a substitution of a formal power series with polynomial coefficients. Our algorithm is definitely more efficient since its cost is dominated by a similar iteration, but the series involved have "constant" coefficients, and whatever the constant field K, operations over K are bound to be more efficient than operations over $K[\zeta]$.

References

[1] Apostol, T. M. *Introduction to Analytic Number Theory.* Springer-Verlag, 1976.

[2] Brent, R. P., and Kung, H. T. Fast algorithms for manipulating formal power series. *Journal of the ACM 25* (1978), 581–595.

[3] Cipolla, M. La determinazione assintotica dell'n^{imo} numero primo. *Rendiconti Acad. Sci. Fis. Mat. Napoli 8* (1902), 132–166.

[4] Comtet, L. Inversion de $y^\alpha e^y$ et $y \log^\alpha y$ au moyen des nombres de Stirling. *Comptes–Rendus de l'Académie des Sciences 270* (1970), 1085–1088.

[5] Comtet, L. *Advanced Combinatorics.* Reidel, Dordrecht, 1974.

[6] De Bruijn, N. G. *Asymptotic Methods in Analysis.* Dover, 1981. A reprint of the third North Holland edition, 1970 (first edition, 1958).

[7] Knuth, D. E. *The Art of Computer Programming*, 2nd ed., vol. 2: Seminumerical Algorithms. Addison-Wesley, 1981.

[8] Massias, J. P., Nicolas, J. L., and Robin, G. Évaluation asymptotique de l'ordre maximum d'un élément du groupe symétrique. *Acta Arithmetica L* (1988), 221–242.

[9] Robin, G. Permanence de relations de récurrence dans certains développements asymptotiques. *Publications de l'Institut Mathématique de Beograd 43*, 57 (1988), 17–25.

Appendices

A A Maple program

The following program implements Theorems 1 and 2.

```
# Compute n terms of the generating function of {P_k(0)}, given α, the power series D in the variable t, and n.
P0:=proc (alpha, d, t, n) local P, u, s;
   if coeff(d,t,0)=0 then ERROR(`Invalid series`,d) fi;
   s:=convert(series(alpha*log(1+t*P)−subs(t=t/(1+t*P),series(log(d),t,n+1)),t,n+1),polynom);
   u:=coeff(s,t,0);
   to n do u:=series(subs(P=u,s),t,n+1) od
end: # P0

# Compute the n first P_k satisfying P'_{n+1} = aP'_n + (bn + c)P_n, given P_0 and the generating series s(t) of {P_k(0)}.
rec:=proc (p0, a, b, c, s, t, n) local P, i;
   P[0]:=p0−coeff(s,t,0);
   for i from 0 to n−1 do P[i+1]:=sort(a*P[i]+(b*i+c)*(int(P[i],t)+coeff(s,t,i)*t),t) od;
   RETURN([seq(P[i]+coeff(s,t,i),i=0..n)])
end: # rec

# Compute n terms of the asymptotic expansion of y, where e^y y^α D(1/y) = x,
# given α and the formal power series D(x).
theorem1:=proc (alpha, d, x, n) local s, i;
   s:=subs(x=log(log(x)),rec(alpha*x−log(coeff(d,x,0)),alpha,−alpha,0,P0(alpha,d,x,n),x,n));
   RETURN(convert([log(x),seq(op(i,s)*log(x)^(1−i),i=1..n+1),O(log(log(x))^(n+1)/log(x)^(n+1))],`+`))
end: # theorem1

# Compute n terms of the asymptotic expansion of log y, where e^y y^α D(1/y) = x,
# given α and the formal power series D(x).
theorem2_log:=proc (alpha, d, x, n) local s, i;
   s:=subs(x=log(log(x)),rec(0,alpha,−alpha,0,series(log(1+x*P0(alpha,d,x,n−1)),x,n+1),x,n));
   RETURN(convert([log(log(x)),seq(op(i,s)*log(x)^(1−i),i=1..n+1),O(log(log(x))^(n+1)/log(x)^(n+1))],`+`))
end: # theorem2_log

# Compute n terms of the asymptotic expansion of e^{βy} y^γ G(1/y), where e^y y^{−α} D(1/y) = x, given α, β, γ,
# and the formal power series D and G.
theorem2_part2:=proc (alpha, beta, gamma, d, g, x, n) local P, s, i;
   s:=subs(x=log(log(x)),rec(coeff(g,x,0),alpha,−alpha,alpha*(alpha*beta+gamma),
         series(subs(P=P0(alpha,d,x,n),exp(beta*(P+log(coeff(d,x,0))))*
         convert(series((1+x*P)^gamma*subs(x=x/(1+x*P),g),x,n+1),polynom)),x,n+1),x,n));
   RETURN((x/coeff(d,x,0))^beta*(log(x))^(alpha*beta+gamma)*convert([
   seq(op(i,s)*log(x)^(1−i),i=1..n+1),O(log(log(x))^(n+1)/log(x)^(n+1))],`+`))
end: # theorem2_part2
```

B Asymptotic expansions

We give the first terms of the expansions mentioned in our corollaries. Since in most of this cases, the first few terms have already been published, these expansions help vindicate both our theorems and our program.

B.1 Maple's W function

> theorem1(1,1,x,7);

$$\ln(x)+\ln(\ln(x))+\frac{\ln(\ln(x))}{\ln(x)}+\frac{-\frac{\ln(\ln(x))^2}{2}+\ln(\ln(x))}{\ln(x)^2}+\frac{\frac{\ln(\ln(x))^3}{3}-3/2\ln(\ln(x))^2+\ln(\ln(x))}{\ln(x)^3}$$
$$+\frac{-\frac{\ln(\ln(x))^4}{4}+11/6\ln(\ln(x))^3-3\ln(\ln(x))^2+\ln(\ln(x))}{\ln(x)^4}$$
$$+\frac{\frac{\ln(\ln(x))^5}{5}-\frac{25}{12}\ln(\ln(x))^4+35/6\ln(\ln(x))^3-5\ln(\ln(x))^2+\ln(\ln(x))}{\ln(x)^5}$$
$$+\frac{-\frac{\ln(\ln(x))^6}{6}+\frac{137}{60}\ln(\ln(x))^5-75/8\ln(\ln(x))^4+85/6\ln(\ln(x))^3-15/2\ln(\ln(x))^2+\ln(\ln(x))}{\ln(x)^6}$$
$$+\frac{\frac{\ln(\ln(x))^7}{7}-\frac{49}{20}\ln(\ln(x))^6+\frac{203}{15}\ln(\ln(x))^5-245/8\ln(\ln(x))^4+175/6\ln(\ln(x))^3-21/2\ln(\ln(x))^2+\ln(\ln(x))}{\ln(x)^7}$$
$$+O\left(\frac{\ln(\ln(x))^8}{\ln(x)^8}\right)$$

B.2 The kth prime number

> $\mathcal{E}$:=convert([seq(i!*k^ i,i=0..10)],'+'):
> theorem2_part2(1,1,0,$\mathcal{E}$,1,k,7);

$$k\ln(k)\left[1+\frac{\ln(\ln(k))-1}{\ln(k)}+\frac{\ln(\ln(k))-2}{\ln(k)^2}+\frac{-\frac{\ln(\ln(k))^2}{2}+3\ln(\ln(k))-11/2}{\ln(k)^3}\right.$$
$$+\frac{\frac{\ln(\ln(k))^3}{3}-7/2\ln(\ln(k))^2+14\ln(\ln(k))-131/6}{\ln(k)^4}$$
$$+\frac{-\frac{\ln(\ln(k))^4}{4}+23/6\ln(\ln(k))^3-49/2\ln(\ln(k))^2+159/2\ln(\ln(k))-\frac{1333}{12}}{\ln(k)^5}$$
$$+\frac{\frac{\ln(\ln(k))^5}{5}-\frac{49}{12}\ln(\ln(k))^4+73/2\ln(\ln(k))^3-367/2\ln(\ln(k))^2+3143/6\ln(\ln(k))-\frac{13589}{20}}{\ln(k)^6}$$
$$+\frac{-\frac{\ln(\ln(k))^6}{6}+\frac{257}{60}\ln(\ln(k))^5-\frac{1193}{24}\ln(\ln(k))^4+1027/3\ln(\ln(k))^3-\frac{17917}{12}\ln(\ln(k))^2+\frac{47053}{12}\ln(\ln(k))-\frac{193223}{40}}{\ln(k)^7}$$
$$\left.+O\left(\frac{\ln(\ln(k))^8}{\ln(k)^8}\right)\right]$$

B.3 The function $\log g(n)$

> theorem2_part2(1,1/2,0,$\mathcal{E}$,1,n,6);

$$\sqrt{n\ln(n)}\left[1+\frac{\frac{\ln(\ln(n))}{2}-1/2}{\ln(n)}+\frac{-\frac{\ln(\ln(n))^2}{8}+3/4\ln(\ln(n))-9/8}{\ln(n)^2}+\frac{\frac{\ln(\ln(n))^3}{16}-\frac{11}{16}\ln(\ln(n))^2+\frac{39}{16}\ln(\ln(n))-\frac{53}{16}}{\ln(n)^3}\right.$$

$$+\frac{-5/128\ln(\ln(n))^4+\frac{61}{96}\ln(\ln(n))^3-\frac{239}{64}\ln(\ln(n))^2+\frac{343}{32}\ln(\ln(n))-\frac{5071}{384}}{\ln(n)^4}$$

$$+\frac{7/256\ln(\ln(n))^5-\frac{457}{768}\ln(\ln(n))^4+\frac{639}{128}\ln(\ln(n))^3-\frac{2879}{128}\ln(\ln(n))^2+\frac{43729}{768}\ln(\ln(n))-\frac{16863}{256}}{\ln(n)^5}$$

$$+\frac{-\frac{21}{1024}\ln(\ln(n))^6+\frac{1441}{2560}\ln(\ln(n))^5-\frac{19081}{3072}\ln(\ln(n))^4+\frac{9915}{256}\ln(\ln(n))^3-\frac{154219}{1024}\ln(\ln(n))^2+\frac{542759}{1536}\ln(\ln(n))-\frac{2012177}{5120}}{\ln(n)^6}$$

$$+O\left(\frac{\ln(\ln(n))^7}{\ln(n)^7}\right)\Bigg]$$

B.4 The function $\omega(g(n))$

> 2*theorem2_part2(1,1/2,-1,$\mathcal{E}$,subs(n=2*n,$\mathcal{E}$),n,6);

$$2\sqrt{\frac{n}{\log n}}\Bigg[1+\frac{-\frac{\ln(\ln(n))}{2}+3/2}{\ln(n)}+\frac{3/8\ln(\ln(n))^2-11/4\ln(\ln(n))+55/8}{\ln(n)^2}$$

$$+\frac{-5/16\ln(\ln(n))^3+\frac{61}{16}\ln(\ln(n))^2-\frac{319}{16}\ln(\ln(n))+\frac{711}{16}}{\ln(n)^3}$$

$$+\frac{\frac{35}{128}\ln(\ln(n))^4-\frac{457}{96}\ln(\ln(n))^3+\frac{2477}{64}\ln(\ln(n))^2-\frac{5615}{32}\ln(\ln(n))+\frac{141937}{384}}{\ln(n)^4}$$

$$+\frac{-\frac{63}{256}\ln(\ln(n))^5+\frac{1441}{256}\ln(\ln(n))^4-\frac{24121}{384}\ln(\ln(n))^3+\frac{55489}{128}\ln(\ln(n))^2-\frac{470731}{256}\ln(\ln(n))+\frac{2894663}{768}}{\ln(n)^5}$$

$$+\frac{\frac{231}{1024}\ln(\ln(n))^6-\frac{16481}{2560}\ln(\ln(n))^5+\frac{282623}{3072}\ln(\ln(n))^4-\frac{658621}{768}\ln(\ln(n))^3+\frac{5621953}{1024}\ln(\ln(n))^2-\frac{34665679}{1536}\ln(\ln(n))+\frac{701392781}{15360}}{\ln(n)^6}$$

$$+O\left(\frac{\ln(\ln(n))^7}{\ln(n)^7}\right)\Bigg]$$

B.5 The general case

Of course, our program is not limited to constant coefficients:
> $D := d_0 + d_1x + d_2x^2 : G := g_0 + g_1x + g_2x^2 :$
> theorem2_part2$(\alpha,\beta,\gamma,D,G,x,2)$;

$$\left(\frac{x}{d_0}\right)^{\beta}\ln(x)^{\alpha\beta+\gamma}\Bigg[g_0+\frac{\alpha\,(\alpha\,\beta+\gamma)\,g_0\,\ln(\ln(x))+g_1-\gamma\,\ln(d_0)\,g_0-\frac{\beta(d_1+\alpha\,\ln(d_0)\,d_0)g_0}{d_0}}{\ln(x)}$$

$$+\frac{1}{\ln(x)^2}\Bigg[\frac{(-\alpha+\alpha\,(\alpha\,\beta+\gamma))\,\alpha\,(\alpha\,\beta+\gamma)\,g_0\ln(\ln(x))^2}{2}$$

$$+\left(\alpha^2\,(\alpha\,\beta+\gamma)\,g_0+(-\alpha+\alpha\,(\alpha\,\beta+\gamma))\left(g_1-\gamma\,\ln(d_0)\,g_0-\frac{\beta\,(d_1+\alpha\,\ln(d_0)\,d_0)\,g_0}{d_0}\right)\right)\ln(\ln(x))$$

$$+\gamma\left(-\frac{d_1}{d_0}-\alpha\,\ln(d_0)\right)g_0+g_1\,\ln(d_0)+g_2-\gamma\,\ln(d_0)\,g_1+\frac{\gamma\ln(d_0)^2\,(\gamma-1)\,g_0}{2}-\frac{\beta\,(d_1+\alpha\,\ln(d_0)\,d_0)\,(g_1-\gamma\,\ln(d_0)\,g_0)}{d_0}$$

$$+\frac{\beta\left(-2\,\alpha\,d_0\,d_1-2\,\alpha^2\,{d_0}^2\,\ln(d_0)-2\,d_1\,\ln(d_0)\,d_0-\alpha\ln(d_0)^2\,{d_0}^2-2\,d_2\,d_0+{d_1}^2+\beta\,{d_1}^2+2\,\beta\,d_1\,\alpha\,\ln(d_0)\,d_0+\beta\,\alpha^2\ln(d_0)^2\,{d_0}^2\right)g_0}{2\,{d_0}^2}\Bigg]$$

$$+O\left(\frac{\ln(\ln(x))^3}{\ln(x)^3}\right)\Bigg]$$